XIV. *On Quaternary Cubics.* *By the Rev.* George Salmon, *D.D., M.R.I.A.* *Communicated by* A. Cayley, *Esq.*

Received June 14,—Read June 21, 1860.

In the following memoir I propose to make an attempt at an enumeration of the invariants, covariants, and contravariants of a quaternary cubic, that is to say, of a homogeneous function of the third order in four variables, which geometrically represents a surface of the third order. This memoir, then, will be in continuation of Mr. Cayley's memoirs on Quantics, wherein a similar analysis is very completely performed for binary quantics as far as the fifth order, and for ternary quantics as far as the third order.

In consequence of the great length of the formulæ when the general equation is used, I work with Mr. Sylvester's canonical form

$$U=ax^3+by^3+cz^3+du^3+ev^3,$$

where the variables are connected by the relation

$$x+y+z+u+v=0.$$

It will be found that the discussion of this form leads to some results resembling those obtained for binary quantics of the fifth order; the one quantic being canonically expressed as the sum of five third powers, the other as the sum of three fifth powers.

Mr. Sylvester has calculated the Hessian of a cubic expressed in the form given above, but in order to obtain with facility new covariants, it is desirable that we should also be in possession of a contravariant. We should then be able to substitute differential symbols for the contragredient variables, and should thus have an operating symbol by the help of which we could derive new covariants from those known already.

Let, then, $\alpha, \beta, \gamma, \delta, \varepsilon$ be contragredient variables; let us suppose that when the original function U is expressed in terms of four independent variables, we have got any contravariant in $\alpha, \beta, \gamma, \delta$; and let us consider what this will become when the function U is expressed in terms of five variables connected by a linear relation. Now the contravariant in question expresses geometrically the condition that the plane $\alpha x+\beta y+\gamma z+\delta u+\varepsilon v$ should possess a certain connexion with the surface represented by U. This plane, expressed in terms of four variables, is

$$(\alpha-\varepsilon)x+(\beta-\varepsilon)y+(\gamma-\varepsilon)z+(\delta-\varepsilon)u;$$

and it is apparent that the contravariant in terms of five letters is derived from that expressed in terms of four letters, by substituting $\alpha-\varepsilon$, $\beta-\varepsilon$, $\gamma-\varepsilon$, $\delta-\varepsilon$ respectively for

$\alpha, \beta, \gamma, \delta$. Every contravariant, then, in five letters will be a function of the differences between $\alpha, \beta, \gamma, \delta, \varepsilon$.

It is easy to show now that $\alpha, \beta, \gamma, \delta, \varepsilon$ are cogredient with $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{du}, \frac{d}{dv}$ respectively. We know, in fact, that when U is expressed in terms of four letters, we may substitute in any contravariant, for $\alpha, \beta, \gamma, \delta$, $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}, \frac{d}{du}$. When, then, U is transformed into a function of five letters, we must substitute for $\frac{d}{dx}$, $\frac{d}{dx}+\frac{d}{dv}\frac{dv}{dx}$, which (in virtue of the relation connecting x, y, z, u, v) is equal to $\frac{d}{dx}-\frac{d}{dv}$; and similarly for $\frac{d}{dy}$, &c. But these are the very transformations by which the contravariant in four letters is expressed in terms of five.

It is equally easy to show that in a covariant we may substitute for x, y, z, u, v, $\frac{d}{d\alpha}, \frac{d}{d\beta}, \frac{d}{d\gamma}, \frac{d}{d\delta}, \frac{d}{d\varepsilon}$.

Let us commence by applying these principles to calculate the discriminant and reciprocant of a quaternary quadric when expressed in the form

$$ax^2+by^2+cz^2+du^2+ev^2.$$

These functions are known when the quadric is expressed in the general form

$$ax^2+by^2+cz^2+du^2$$
$$+2lyz+2mzx+2nxy+2pxu+2qyu+2rzu.$$

In this case the discriminant is—

(No. 1.)

$$abcd-(adl^2+bdm^2+cdn^2+bcp^2+caq^2+abr^2)$$
$$+2(alqr+bmpr+cnpq+dlmn)$$
$$+l^2p^2+m^2q^2+n^2r^2-2mnqr-2nlrp-2lmpq.$$

The reciprocant (which is also called the bordered Hessian, since it is obtained by bordering with contragredient variables the determinant which expresses the Hessian) is—

(No. 2.)

$$\alpha^2(bcd-br^2-cq^2-dl^2+2lqr)$$
$$+\beta^2(cda-cp^2-dm^2-ar^2+2pmr)$$
$$+\gamma^2(dab-dn^2-aq^2-bp^2+2nqp)$$
$$+\delta^2(abc-al^2-bm^2-cn^2+2lmn)$$
$$+2\beta\gamma(-adl+aqr+dmn+lp^2-nrp-mpq)$$
$$+2\gamma\alpha(-bdm+bpr+dln+mq^2-nqr-lpq)$$
$$+2\alpha\beta(-cdn+cpq+dlm+nr^2-mqr-lrp)$$
$$+2\alpha\delta(-bcp+bmr+cnq+l^2p-nlr-lmq)$$
$$+2\beta\delta(-caq+alr+cnp+m^2q-mnr-lmp)$$
$$+2\gamma\delta(-abr+alq+bmp+n^2r-mnq-nlp).$$

Let us now examine what the latter function becomes when the given quantic is in the form

$$ax^2+by^2+cz^2+du^2+ev^2.$$

Eliminating v by the help of the relation connecting x, y, z, u, v, the coefficients of x^2, y^2, z^2, u^2 become $a+e$, $b+e$, $c+e$, $d+e$ respectively, while every other coefficient becomes e. Substituting these values in No. 2, the contravariant is

$$\begin{aligned}&\alpha^2\{bcd+(bc+cd+db)e\}\\&+\beta^2\{cda+(da+ac+cd)e\}\\&+\gamma^2\{dab+(ab+bd+da)e\}\\&+\delta^2\{abc+(bc+ca+ab)e\}\\&-2e\{ad\beta\gamma+bd\gamma\alpha+cd\alpha\beta+bc\alpha\delta+ca\beta\delta+ab\gamma\delta\}.\end{aligned}$$

Lastly, write in the form just found $\alpha-\varepsilon$, $\beta-\varepsilon$, $\gamma-\varepsilon$, $\delta-\varepsilon$ for α, β, γ, δ, and it becomes

$$bcd(\alpha-\varepsilon)^2+cda(\beta-\varepsilon)^2+dab(\gamma-\varepsilon)^2+abc(\delta-\varepsilon)^2+bce(\alpha-\delta)^2+cae(\beta-\delta)^2$$
$$+abe(\gamma-\delta)^2+ade(\beta-\gamma)^2+bde(\alpha-\gamma)^2+cde(\alpha-\beta)^2,$$

or, as it may be written, (No. 2 bis)

$$\Sigma cde(\alpha-\beta)^2.$$

I have given this work at length, in order fully to illustrate the process employed in other cases. The discriminant may be obtained either by substituting in No. 1, $a+e$, $b+e$, $c+e$, $d+e$ for a, b, c, d, or by substituting in No. 2 bis, differential symbols for α, β, γ, δ, ε, and operating on U. In either case we get—

(No. 1 bis)

$$bcde+cdea+deab+eabc+abcd,$$

or

$$\Sigma abcd.$$

If the given quadric had been given as a function in its most general form of five variables connected by the relation $x+y+z+u+v$, its discriminant would have been got by taking the bordered Hessian of the quadric (the variables being considered as independent), and then writing $\alpha=\beta=\gamma=\delta=\varepsilon=1$. And in general it is evident that any invariant of a function of n variables connected by the relation

$$\alpha x+\beta y+\gamma z+\delta u+\varepsilon v+\&c.=0,$$

is a contravariant of the same form considered as a function of n independent variables, α, β, γ, &c. being the contragredient variables.

The reciprocant and discriminant are the only functions concomitant to a quadric. Let us proceed, then, to the quaternary cubic

$$U=ax^3+by^3+cz^3+du^3+ev^3.$$

Its Hessian is the discriminant of any polar quadric,

$$axx'^2+byy'^2+czz'^2+duu'^2+evv'^2,$$

therefore, by No. 1 bis, is immediately, as Mr. SYLVESTER has also proved,

$$bcdeyzuv + cdeazuvx + deabuvxy + eabcvxyz + abcdxyzu,$$

or

$$\Sigma abcdxyzu.$$

We also get immediately from No. 2 bis a mixed concomitant, viz. the bordered Hessian, which may be written

$$\Sigma cdezuv(\alpha-\beta)^2.$$

If in this last we substitute differentials for α, β, &c. and operate on U, we get the Hessian. If we operate on the Hessian, we get a covariant of the fifth order, to which I shall afterwards refer as Φ,

$$\Phi = abcde\Sigma abx^2y^2z.$$

The reciprocant of the given quantic being, as is known, of the form $64S^3=T^2$, presents us with two other contravariants, S and T, the consideration of which is very fertile in results. If the quantic were to reduce itself to the form

$$ax^3+by^3+cz^3+du^3,$$

it is easy to see that the reciprocant reduces to

$$64(abcd\alpha\beta\gamma\delta)^3 = \{\Sigma b^2c^2d^2\alpha^6 - 2\Sigma abc^2d^2\alpha^3\beta^3\}^2.$$

From this, then, we can readily anticipate the form of S in the general case, which, however, I have verified independently. It is of the fourth order in the variables and in the letters, and is

$$S = \Sigma abcd(\alpha-\varepsilon)(\beta-\varepsilon)(\gamma-\varepsilon)(\delta-\varepsilon);$$

that is to say,

$$\begin{aligned} &bcde(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon)+cdea(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\varepsilon)\\ &\quad+deab(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\varepsilon)+eabc(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\varepsilon)\\ &\quad+abcd(\varepsilon-\alpha)(\varepsilon-\beta)(\varepsilon-\gamma)(\varepsilon-\delta).\end{aligned}$$

From this we can derive several other concomitant functions. For instance, operating on this with U, we expect to get a contravariant of the first degree in the variables, and the fifth in the coefficients, but this will be found to vanish identically. Operating with S on the Hessian, we get the simplest invariant of the cubic, to which I shall refer as Invariant A,

$$A = \Sigma b^2c^2d^2e^2 - 2abcde\Sigma abc.$$

Symbolically A is $(1235)(1246)(1347)(2348)(5678)^2$.

Again operating with S on U^2, we get the simplest covariant quadric which is of the sixth order in the coefficients, and is

$$abcde(ax^2+by^2+cz^2+du^2+ev^2).$$

Symbolically this covariant is expressed (1234)(1235)(1456)(2456). Again operating with this on S, we get a contravariant quadric of the tenth order in the coefficients; and again operating with this last on U, we obtain a covariant of the first order in the variables and the eleventh in the coefficients. But it being sufficiently obvious that new invariants, covariants, and contravariants can be formed *ad libitum* by the process already explained of combining a covariant and contravariant already found, I consider it needless to enter into further detail as to the steps by which particular covariants are obtained, and proceed to form into tables the most important results.

I write, for brevity,

$$a+b+c+d+e=p,$$
$$ab+ac+ad+ae+bc+bd+be+cd+ce+de=q,$$
$$cde+bde+bce+bcd+ade+ace+acd+abe+abd+abc=r,$$
$$bcde+cdea+deab+eabc+abcd=s,$$
$$abcde=t.$$

I commence with the invariants of the cubic, which appear to be all reducible to the five following fundamental invariants of the 8th, 16th, 24th, 32nd, and 40th orders respectively:—

$$A=s^2-4rt.$$
$$B=t^3p.$$
$$C=t^4s.$$
$$D=t^6q.$$
$$E=t^8.$$

Whence also $$C^2-AE=4t^9r.$$

Since any invariant must be a symmetric function of a, b, c, d, e, it can be expressed in terms of p, q, r, s, t, and therefore in terms of A, B, C, D, E. We can form, however, precisely as in the case of binary quintics, skew invariants which cannot be expressed rationally in terms of the five fundamental invariants, but whose squares can be expressed as rational functions of these quantities. The simplest invariant of this kind is of the hundredth degree in the coefficients, and for the canonical form is t^{18} multiplied by the product of all the differences between any two of the quantities a, b, c, d, e. The following is the expression for the square of this invariant F in terms of the simpler invariants:—

$$256F^2=800000\,E^5+240000\,E^4DA-640000\,E^4CB+36000\,E^4CA^2-128000\,E^4B^2A$$
$$+3600\,E^4BA^3-27\,E^4A^5+576000\,E^3D^2B+13200\,E^3D^2A^2+272000\,E^3DC^2$$
$$+131200\,E^3DCBA+2520\,E^3DCA^3-409600\,E^3DB^3+8960\,E^3DB^2A^2-72\,E^3DBA^4$$
$$+30400\,E^3C^3A+115200\,E^3C^2B^2+5520\,E^3C^2BA^2+108\,E^3C^2A^4-10240\,E^3CB^3A$$
$$-96\,E^3CB^2A^3+65536\,E^3B^5-2048\,E^3B^4A^2+16\,E^3B^3A^4-230400\,E^2D^3C$$
$$+40960\,E^2D^3AB-64\,E^2D^3A^3-62240\,E^2D^2C^2A+261120\,E^2D^2CB^2$$
$$+5696\,E^2D^2CBA^2-90240\,E^2DC^3B-9216\,E^2D^2B^3A+16\,E^2D^2B^2A^3-5256\,E^2DC^3A^2$$

$$+29824\,E^2DC^2B^2A+216\,E^2DC^2BA^3-49152\,E^2DCB^4-1280\,E^2DCB^3A^2-864\,E^2C^5$$
$$-9552\,E^2C^4BA-162\,E^2C^4A^3+192\,E^2C^3B^2A^2+1024\,E^2C^3B^3-5120\,E^2C^2B^4A$$
$$-48\,E^2C^2B^3A^3+27648\,ED^5-6912\,ED^4B^2+4608\,ED^4CA-34816\,ED^3C^2B$$
$$+128\,ED^3C^2A^2-1152\,ED^3CB^2A+16272\,ED^2C^4-6272\,ED^2C^3BA+7680ED^2C^2B^3$$
$$+2952\,EDC^5A-32\,ED^2C^2B^2A^2-1920\,EDC^4B^2-216\,EDC^4BA^2+1408\,EDC^3B^3A$$
$$+432\,EC^6B+108\,EC^6A^2-96\,EC^5B^2A+256\,EC^4B^4+48\,EC^4B^3A^2-512\,D^4C^3$$
$$-64\,D^3C^4A+128\,D^3C^3B^2+576\,D^2C^5B+16D^2C^4B^2A+72\,DC^6BA-128\,DC^5B^3$$
$$-216\,DC^7-27\,C^8A-16\,C^6B^3A.$$

I next proceed to show how the discriminant of the surface is expressed in terms of the fundamental invariants A, B, C, D. And it will throw some light on the investigation if I first give the solution of the corresponding problem for a ternary cubic expressed in the form

$$ax^3+by^3+cz^3+du^3,\ \text{where}\ x+y+z+u=0.$$

The following is a Table of the principal concomitants of this form:—

ARONHOLD's invariants:

$$S=abcd,$$
$$T=a^2b^2c^2+b^2c^2d^2+c^2d^2a^2+d^2a^2b^2-2abcd(ab+ac+ad+bc+cd+db).$$

The Hessian:

$$H=bcdyzu+cdazux+dabuxy+abcxyz.$$

It is obvious that the points xy, zu; xz, yu; xu, yz are pairs of corresponding points on the Hessian. The evectant of S, which Mr. CAYLEY calls the Pippian, is

$$bcd(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)+cda(\beta-\alpha)(\beta-\gamma)(\beta-\delta)+dab(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)$$
$$+abc(\delta-\alpha)(\delta-\beta)(\delta-\gamma),$$

from which form it can easily be deduced that the lines joining corresponding points on the Hessian touch the curve represented by this evectant, as do also the two lines into which the polar conic of any point on the Hessian breaks up.

The first evectant of T (called by Mr. CAYLEY the Quippian) is

$$\Sigma c^2d^2(a-b)(\alpha-\beta)^3-\Sigma d^2abc(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta);$$

and the second evectant of T, that is to say the Reciprocant, is

$$\Sigma c^2d^2(\alpha-\beta)^6-2\Sigma d^2bc(\alpha-\beta)^3(\alpha-\gamma)^3$$
$$+2abcd\{(\alpha-\gamma)(\beta-\delta)-(\alpha-\delta)(\gamma-\beta)\}\{(\alpha-\delta)(\gamma-\beta)-(\alpha-\beta)(\delta-\gamma)\}$$
$$\{(\alpha-\beta)(\delta-\gamma)-(\alpha-\gamma)(\beta-\delta)\}.$$

To return now to the problem of finding the discriminant of this plane cubic. This is obtained by eliminating the variables between the three polar conics got by differentiating the equation with regard to x, y, z respectively; viz.

$$ax^2-du^2,\quad by^2-du^2,\quad cz^2-du^2.$$

Each of these breaks up into factors: choosing any set of these factors, the resultant of the four factors comes out easily in the form of the determinant

$$\begin{vmatrix} \sqrt{a}+\sqrt{d}, & \sqrt{d} & , \sqrt{d} \\ \sqrt{d}, & \sqrt{b}+\sqrt{d}, & \sqrt{d} \\ \sqrt{d}, & \sqrt{d} & , \sqrt{c}+\sqrt{d}; \end{vmatrix}$$

that is to say,

$$\sqrt{bcd}+\sqrt{cda}+\sqrt{dab}+\sqrt{abc}.$$

We are then to multiply together the eight results obtained by giving all possible variations of signs to the radicals, and the product will evidently be the same as the result of clearing of radicals the equation

$$\sqrt{bcd}+\sqrt{cda}+\sqrt{dab}+\sqrt{abc}=0;$$

that is to say,

$$\{b^2c^2d^2+c^2d^2e^2+d^2a^2b^2+a^2b^2c^2-2abcd(ab+ac+ad+bc+cd+db)\}^2=64a^3b^3c^3d^3,$$

or

$$T^2=64\,S^3.$$

Evidently then, by precisely the same process, the discriminant of the surface

$$ax^3+by^3+cz^3+du^3+ev^3$$

is the result of clearing of radicals the equation

$$\sqrt{bcde}+\sqrt{cdea}+\sqrt{deab}+\sqrt{eabc}+\sqrt{abcd}=0;$$

and on performing the operation the result is found to be

$$\{A^2-64B\}^2=16384\{D+2AC\};$$

where A, B, C, D are the fundamental invariants described above.

It is evident, in like manner, that if it be required to find the result of elimination between three ternary quadrics, since we can write these functions in the form

$$\begin{aligned} a\,x^2+by^2+cz^2+du^2, \\ a'\,x^2+b'y^2+c'z^2+d'u^2, \\ a''x^2+b''y^2+c''z^2+d''u^2, \end{aligned}$$

we can reduce these to the form

$$\alpha x^2=\beta y^2=\gamma z^2=\delta u^2;$$

that therefore, as before, the eliminant can be expressed as the result of clearing of radicals an equation of the form

$$\sqrt{\beta\gamma\delta}+\sqrt{\gamma\delta\alpha}+\sqrt{\alpha\beta\delta}+\sqrt{\alpha\beta\gamma}=0,$$

and therefore, as Mr. Sylvester first pointed out, the eliminant is of the form

$$T^2=64\,S.$$

But in this case S is not a perfect cube, as it turns out in the problem of finding the discriminant of a cubic. In fact, in this case S is the product of the four determinants of the system

$$\left\|\begin{matrix} a\,, & b\,, & c\,, & d \\ a'\,, & b'\,, & c'\,, & d' \\ a''\,, & b''\,, & c''\,, & d'' \end{matrix}\right\|$$

and therefore its vanishing expresses the condition that some one of the quadrics of the system $\lambda U+\mu U'+\nu U''$ shall be a perfect square; or geometrically, it expresses the condition that it shall be possible, through the intersection of two conics of the system, to draw a conic having double contact with the third. In this case the Jacobian of the system breaks up into a right line and conic.

But we cannot, in like manner, conclude that the result of elimination between four quaternary quadrics is expressible as the result of clearing of radicals the sum of five square roots, because the quadrics cannot in general be expressed as four functions each of the form

$$ax^2+by^2+cz^2+du^2+ev^2.$$

We proceed now, from the consideration of the invariants of a quaternary cubic, to that of its covariants. To the number of these, however, there is, as far as I am aware, no limit, and I must therefore confine myself to enumerating some of the simplest and most important. Of covariants of the first order in the variables, the following may be regarded as the four fundamental forms. The simplest, L, of the eleventh order in the coefficients,

$$L=a^2b^2c^2d^2e^2\{ax+by+cz+du+ev\}\,;$$

of the nineteenth order in the coefficients,

$$L'=a^3b^3c^3d^3e^3\{bcdex+cdeay+deabz+eabcu+abcdv\}\,;$$

of the twenty-seventh order,

$$L''=a^5b^5c^5d^5e^5\{a^2x+b^2y+c^2z+d^2u+e^2v\}.$$

There is no covariant of the first order in the variables and 35th in the coefficients which may not be expressed in terms of the preceding three and the fundamental invariants; but we have a fourth of the 43rd order, viz.

$$L'''=a^8b^8c^8d^8e^8\{a^3x+b^3y+c^3z+d^3u+e^3v\}.$$

Every other covariant can be expressed in terms of these four. The resultant of these four covariants, that is to say, the condition that they shall represent four planes meeting in a point, is the invariant F of the 100th degree already noticed. The cubic itself can be expressed as a function of L, L′, L″, L‴ having invariant coefficients, which expression M. HERMITE calls the form-type of the function. I have not thought it worth while to undertake the labour of finding the actual values of the coefficients of the function so transformed.

The simplest linear contravariant is of the 13th degree, that of the 5th vanishing, as before mentioned. It is

$$abcde\Sigma(a-b)(\alpha-\beta)\{(a+b)c^2d^2e^2-abcde(cd+de+ec)\}.$$

The coefficient of α in this is

$$abcde\{\mathrm{A}-5b^2c^2d^2e^2+5abcde(bcd+bce+bcd+cde)\},$$

A being the simplest invariant.

The next linear contravariant is of the 21st degree,

$$a^4b^4c^4d^4e^4\Sigma(a-b)(\alpha-\beta).$$

The next is of the 29th degree,

$$a^5b^5c^5d^5e^5\Sigma cde(a-b)(\alpha-\beta),$$

and so on.

The simplest covariant quadric is of the 6th degree,

$$abcde(ax^2+by^2+cz^2+du^2+ev^2).$$

The next of the 14th degree,

$$a^2b^2c^2d^2e^2\Sigma ab(cd+de+ec)xy,$$

which may also be written

$$a^2b^2c^2d^2e^2\Sigma(bcde-abcd-abce-abde-acde)x^2.$$

The next of the 22nd degree, may most simply be written

$$a^4b^4c^4d^4e^4(a^2x^2+b^2y^2+c^2z^2+d^2u^2+e^2v^2).$$

That of the 30th is

$$a^6b^6c^6d^6e^6(x^2+y^2+z^2+u^2+v^2),$$

and so on.

The contravariant quadrics are—

10th order; $a^2b^2c^2d^2e^2\Sigma(\alpha-\beta)^2$.

18th order; $a^3b^3c^3d^3e^3\Sigma cde(\alpha-\beta)^2$, &c.

The covariant cubics are—

9th order; $\Sigma cde(a+b)abcde\ zuv$

17th order; $a^3b^3c^3d^3e^3(a^3x^3+b^3y^3+c^3z^3+d^3u^3+e^3v^3)$.

This last covariant is important, because we can immediately deduce from it, that the operation

$$t^3\left(a^2\frac{d}{da}+b^2\frac{d}{db}+c^2\frac{d}{dc}+d^2\frac{d}{dd}+e^2\frac{d}{de}\right)$$

performed on any invariant or covariant of the cubic

$$ax^3+by^3+cz^3+du^3+ev^3$$

gives rise to a new invariant or covariant of the cubic, and of the degree sixteen higher in the coefficients. It is on this account that I think it enough to write down two covariants of each degree in the variables.

The simplest contravariant of the third order is the evectant of the invariant A, which is of the seventh order in the coefficients. It is

$$\Sigma c^2d^2e^2(a-b)(\alpha-\beta)^3-\Sigma abcd^2e^2(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta).$$

The next is of the fifteenth,

$$\Sigma a^2b^2c^2d^2e^2(bcde)(a-b)(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon).$$

After the Hessian, the simplest covariant quartic is

$$a^2b^2c^2d^2e^2(a^2x^4+b^2y^4+c^2z^4+d^2u^4+e^2v^4).$$

We have already mentioned the simplest contravariant quartic, viz.

$$S=\Sigma bcde(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\varepsilon).$$

The reciprocant of the cubic is $64S^3=T^2$, where T is the contravariant sextic,

$$\begin{aligned}T=&\Sigma c^2d^2e^2(\alpha-\beta)^6-2\Sigma bcd^2e^2(\alpha-\beta)^3(\alpha-\gamma)^3\\&+2\Sigma abcde^2\{(\alpha-\gamma)(\beta-\delta)-(\alpha-\delta)(\gamma-\beta)\}\{(\alpha-\delta)(\gamma-\beta)-(\alpha-\beta)(\delta-\gamma)\}\\&\{(\alpha-\beta)(\delta-\gamma)-(\alpha-\gamma)(\beta-\delta)\}.\end{aligned}$$

I have already mentioned the covariant quintic

$$\Phi=abcde\Sigma(abx^2y^2z),$$

which, however, may be replaced by

$$abcde(a^2x^5+b^2y^5+c^2z^5+d^2u^5+e^2v^5),$$

the two differing only by the product of the original cubic and the simplest covariant quadric.

The simplest covariant of the fifteenth order in the coefficients and fifth in the variables, is what may be called the canonizant $a^3b^3c^3d^3e^3xyzuv$, since it represents the five planes of the canonical form.

The only covariant of higher order which we shall notice is that of the ninth order, which we shall call the covariant Θ, viz. the determinant,

$$\begin{vmatrix} \frac{d^2U}{dx^2}, & \frac{d^2U}{dxdy}, & \frac{d^2U}{dxdz}, & \frac{d^2U}{dxd\omega}, & \frac{dH}{dx} \\ \frac{d^2U}{dydx}, & \frac{d^2U}{dy^2}, & \frac{d^2U}{dydz}, & \frac{d^2U}{dyd\omega}, & \frac{dH}{dy} \\ \frac{d^2U}{dzdx}, & \frac{d^2U}{dzdy}, & \frac{d^2U}{dz^2}, & \frac{d^2U}{dzd\omega}, & \frac{dH}{dz} \\ \frac{d^2U}{d\omega dx}, & \frac{d^2U}{d\omega dy}, & \frac{d^2U}{d\omega dy}, & \frac{d^2U}{d\omega^2}, & \frac{dH}{d\omega} \\ \frac{dH}{dx}, & \frac{dH}{dy}, & \frac{dH}{dz}, & \frac{dH}{d\omega}. & \end{vmatrix}$$

The importance of this covariant consists in this, that the twenty-seven right lines on the cubic are determined as the intersection with the given cubic of the surface of the ninth order, $\Theta=4H\Phi$, where Φ is the covariant of the fifth order just mentioned. This can be verified with little labour by taking the general equation of a surface passing through the line xy and examining the terms in Θ, H, and Φ, which do not contain x or y. I have already noticed* that the existence of these twenty-seven right lines may be deduced as a particular case of the following theorem, that on a surface of the nth degree there is a locus of points at each of which it is possible to draw a line to meet the surface in four consecutive points, this locus being the intersection of the given surface with a surface of the degree $11n-24$. I gave the equation of this surface, but in a very inconvenient form†. But I have now to add a simpler form of the general equation of this surface, viz.

$$\Theta=4\mathrm{H}(\Phi+a\Psi),$$

where Θ, H, Φ have the meaning already explained; a is a numerical multiplier $\left(\frac{(n-2)(n-3)}{1.2}\right)$, and Ψ is a covariant obtained as follows:—Let Θ when expanded be

$$\mathrm{A}\left(\frac{d\mathrm{H}}{dx}\right)^2+\mathrm{B}\left(\frac{d\mathrm{H}}{dy}\right)^2+\text{ \&c.}+2\mathrm{A}'\frac{d\mathrm{H}}{dx}\frac{d\mathrm{H}}{dy}+\text{ \&c.},$$

then

$$\Psi=\left\{\mathrm{A}\frac{d^2}{dx^2}+\mathrm{B}\frac{d^2}{dy^2}+\text{ \&c. }+2\mathrm{A}'\frac{d^2}{dxdy}+\text{\&c.}\right\}^2\mathrm{U};$$

that is to say, Ψ is the result of squaring the bordered Hessian, introducing differential symbols instead of the contravariant variables, and operating with the result on U. In the case of the cubic Ψ evidently vanishes.

* Cambridge and Dublin Mathematical Journal, vol. iv. p. 258.

† Quarterly Journal of Mathematics, vol. i. p. 336.